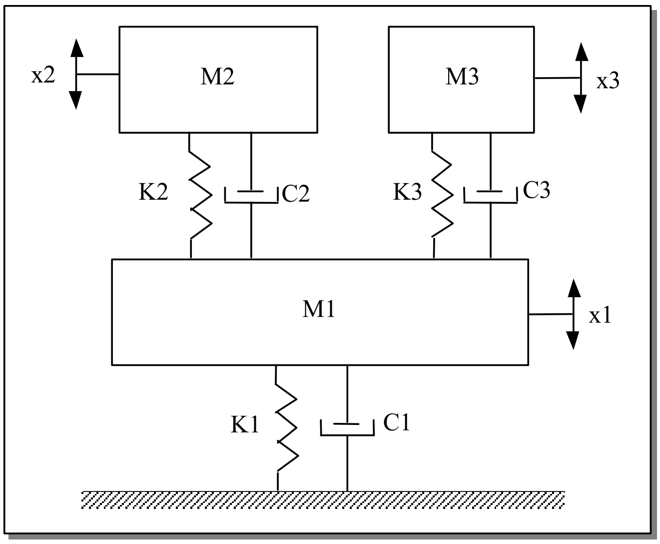
Signals & Systems

Laboratory no. 6 – Linear algebra

EXAMPLE 1

Eigenvalue problem.

Let us consider the system from the figure below.



Please find the system modal parameters (natural frequencies, damping ratios and modal vectors) for the following data:

Masses:

M1 = 5 [kg]

M2 = 2 [kg]

M3 = 1 [kg]

Damping coefficients:

C1 = 10 [N s / m]

C2 = 5 [N s / m]

C3 = 6 [N s / m]

Stiffness coefficients:

K1 = 60000 [N / m]

K2 = 12000 [N / m]

K3 = 10000 [N / m]

The first step is to determine the equation of motion of the system and save them in the matrix form:

$$[M]\cdot\ddot{x}+[C]\cdot\dot{x}+[K]\cdot x=F$$

Next we Laplace transform the above equation:

$$\left(s^2\cdot[M]+s\cdot[C]+[K]\right)\cdot[X(s)]=[F(s)]$$

If we assume that the vector F is zero (free vibrations system) we are dealing with a typical homogeneous system of equations. To find its nontrivial solution we follow the transformation

$$(s\cdot[M]-s\cdot[M])=0$$
$$(s\cdot[A]+[B])\cdot[Y]=0\Rightarrow(s\cdot[A]+[B])=0$$

where:

$$[A]=\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix},\ [B]=\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix},\ [Y]=\begin{bmatrix}s\cdot[X]\\ [X]\end{bmatrix}$$

To solve above equation we use generalized eigenvalue problem in the form of the formula:

$$A\cdot\phi=\lambda\cdot B\cdot\phi$$

To speed up our calculations the whole analysis will be carried out using Matlab software. To solve the eigenvalue problem in Matlab we will use the commend `eig`.
Below is a fragment of a program for resolving the issue:

```
% Liczba stopni swobody
n=3;
% Masy w układzie
m1=5;
m2=2;
m3=1;
% Współczynniki tłumienia
al1 = 10;
al2 = 5;
al3 = 6;
% Współczynniki sztywności
k1 = 60000;
k2 = 12000;
k3 = 10000;
% Macierze współczynników
% Mas
M = [m1,  0,  0;
      0, m2,  0;
      0,  0, m3];
% Ws. tłumienia
C = [al1+al2+al3, -al2, -al3;
            -al2,  al2,    0;
            -al3,    0,  al3];
```

```
% Ws. sztywności
K = [k1+k2+k3, -k2, -k3;
          -k2,  k2,   0;
          -k3,   0,  k3];
% Budowanie macierzy do równań stanu w oparciu o wzor (8)
ZER = zeros(size(M));
A = [ZER,M;M,C];
B = [-M,ZER;ZER,K];
% Rozwiązanie uogólnionego zagadnienia wlasnego
[PHI,LAMBDA]=eig(-B,A);
% Czestotliwosci drgan wlasnych tlumionych [Hz]
WD=imag(diag(LAMBDA))/2/pi;
% Czestotliwosci drgan wlasnych [Hz]
WW=sqrt(imag(diag(LAMBDA)).^2+real(diag(LAMBDA)).^2)/2/pi;
% Tlumienie modalne
KSI=-real(diag(LAMBDA))./sqrt(imag(diag(LAMBDA)).^2+real(diag(LAMBDA)).^2);
```

This algorithm led us to the designation of natural frequencies and associated modal damping ratios and mode shapes. Based on these data it is possible to synthesize the frequency characteristics in form of frequency response functions. This synthesis is proceed according to the following formula:

$$H_{ij}(\omega)=\sum_{k=1}^{N}\left(\frac{r_{ijk}}{j\omega-\lambda_k}+\frac{r_{ijk}^*}{j\omega-\lambda_k^*}\right)$$

where: $H_{ij}(\omega)$ – FRF between response measured in i and input in j,

N – number of natural frequencies in the considered frequency band

r_{ijk} – modal residua of k-th mode shape, $r_{ijk}=a_k\phi_{ik}\phi_{jk}$

λ_k – pole value for k-th mode shape

* - denotes conjugated number

Below is a fragment of a program for synthesis of the FRFs:

```
% Estymacja współczynników skalujących
AAA=PHI'*A*PHI;
for a=1:n
    AN(:,2*a-1)=AAA(:,2*a);
    AN(:,2*a)=AAA(:,2*a-1);
end;
QQ=inv(AN);
Q=diag(QQ);
% Synteza WFP zgodnie ze wzorem (9)
for c=1:3
    jj=[1:3];
    f=[0:0.25:40];
    for b=1:length(f)
        for a=1:n
            htemp=0;
            for r=1:2*n
                htemp=htemp+((Q(r)*PHI(n+a,r)*
PHI(n+jj(c),r))/(i*f(b)*2*pi- LAMBDA(r,r)));
            end;
            H{a,c}(b)=htemp*(-1)*(f(b)*2*pi)^2;
        end;
    end;
end;
plot(f,abs(H{1,3}))
```

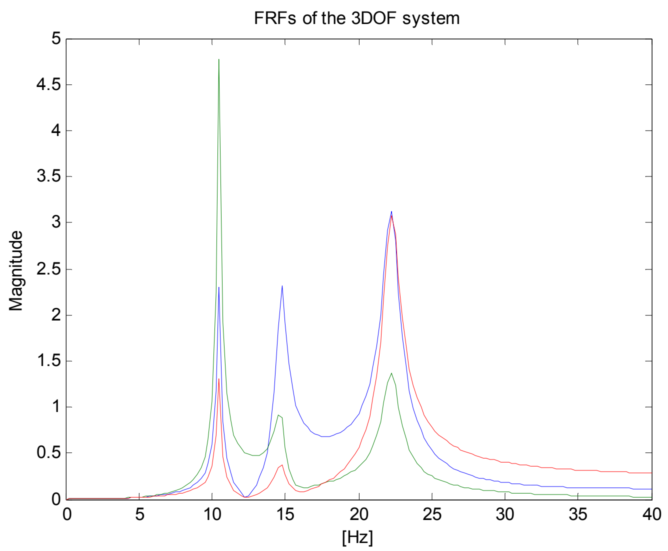
Execution of both parts of the code gives the following results:
Natural frequencies:

```
10.4590 Hz
14.6984 Hz
22.2519 Hz
```

Damping factors:

```
0.0105
0.0233
0.0251
```

FRFs:



EXAMPLE 2

Consider the dynamic system as in Example 1. Suppose that the system is excited in mass M1 by a random force. For some reason, this force can not be measured, so it must be identified on the basis of the known model of the system in form of *H* (FRFs) matrix and the response of the system *x* to force to be identified. The system model is given in the form of synthesized FRFs from the previous example. The system is therefore described by the following equation:

$$x = H \cdot f$$

This type of problem is called a foreword problem. Task of force *f* identification is called the inverse problem of identification and is given by the following formula:

$$f = H^{-1} \cdot x$$

For the process of force identification we will use 3 responses measured on masses 1, 2 and 3, and identified the force is only one, so the size of the *H* matrix is {3 x 1}. Therefore we can not invert the matrix *H,* such an operation is possible only for square matrices. It therefore needs to be pseudoinverted based on the singular value decomposition:

$$H = U \cdot \Sigma \cdot V^T$$

where: $\Sigma = diag(\sigma_1,.., \sigma_r, 0,..., 0)$ of size n x m,

$\sigma_1 \geq ... \geq \sigma_r > 0$,

r – rank of the H matrix,

U, V – orthogonal matrices of sizes respectively n x n and m x m.

Substitutimg the above equation to the inverse problem we will obtein:

$$\Sigma \cdot \xi = \beta$$

where:

$$\xi = V^T \cdot f$$

$$\beta = U^T \cdot x$$

Above equation can be further expressed as a system of equations:

$$\begin{cases} \sigma_1 \cdot \xi_1 = \beta_1 \\ \vdots \\ \sigma_r \cdot \xi_r = \beta_r \\ 0 = \beta_{r+1} \\ \vdots \\ 0 = \beta_n \end{cases}$$

From the above system of equations we can uniquely determine first r components of the vector ξ. So we can replace the inverse peoblem equation with the equation given below:

$$f = V \cdot \xi$$

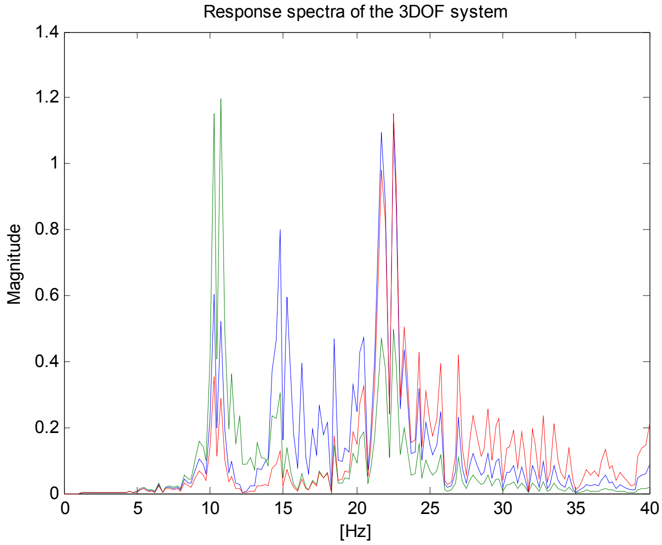
To speed up our calculations the whole analysis will be carried out using Matlab software. To solve the singular value decomposition in Matlab we will use the commend `svd`.
Below is a fragment of a program for resolving the issue:

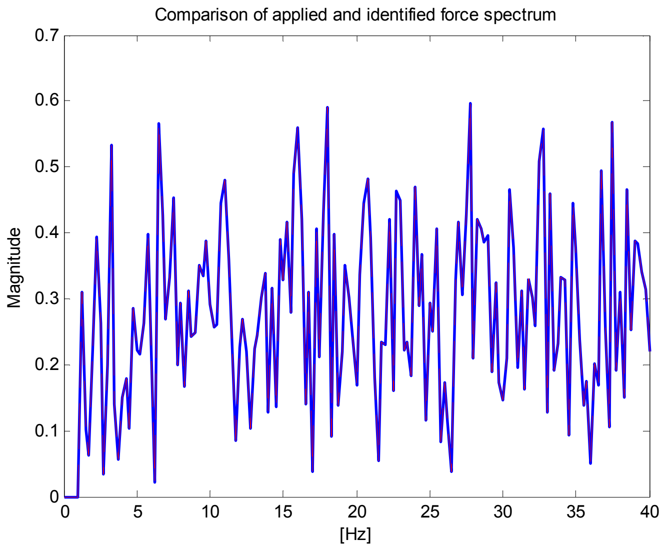
```
% definicja sygnału wymuszenia
% definicja wektora czasu
t=[0:1/80:10];
% definicja wektora siły o charakterze losowym
f_zadane_1=10*rand(size(t));
% rozdzielczosc czestotliwosciowa
nn=round(80/0.25);
% transformata Fouriera
f_zadane_1_freq=(fft(f_zadane_1,nn)*2)/nn;
dl=length(f_zadane_1_freq);
f_zadane_freq{1}=f_zadane_1_freq(1:ceil(dl/2)+1);
f_zadane_freq{1}(1:5)=0;
% wyliczenie odpowiedzi na wymuszenie w masie nr 1
frfs=H(:,1);
for a=1:length(f)
    tfrfs=[];
    tforces=[];
    for b=1:length(f_zadane_freq)
        tforces(b)=f_zadane_freq{b}(a);
    end;
```

```
    for c=1:size(frfs,2)
        for b=1:size(frfs,1)
            tfrfs(b,c)=frfs{b,c}(a);
        end;
    end;
    tresps=tfrfs*tforces';
    for b=1:size(frfs,1)
        resp{b}(a)=tresps(b);
    end;
end;
% wyliczenie wektora siły wymuszającej f
for a=1:length(resp{1})

    % tworzenie tymczasowego wektora widm odpowiedzi i WFP dla kolejnych
    % czestotliwosci
    tresps=[];
    for b=1:length(resp)
        tresps(b)=resp{b}(a);
    end;
    for c=1:size(frfs,2)
        for b=1:size(frfs,1)
            tfrfs(b,c)=frfs{b,c}(a);
        end;
    end;
    tresps=tresps.';
    % rozwiazanie rownania f=(H^-1)*p metoda rozkladu na wartosci
szczegolne
    [u,sig,v]=svd(tfrfs);
    r=rank(tfrfs);
    beta=u'*tresps;
    for b=1:r
        en(b)=beta(b)/sig(b,b);
    end;
    for b=r+1:size(v,1)
        e2(b)=0;
    end;
    if r>0
        esizen=size(en,2);
        esize2=size(e2,2);
        en=[en,zeros(1,size(v,1)-esizen)];
        e2=[zeros(1,size(v,1)-esize2),e2];
        x1=v*en';
        x2=v*e2';
        ftemp=x1+x2;
    else
        esize2=size(e2,2);
        e2=[zeros(1,size(v,1)-esize2),e2];
        x2=v*e2';
        ftemp=x2;
    end;
    for b=1:length(ftemp)
        Identforce{b}(a)=ftemp(b);
    end;
end;
```

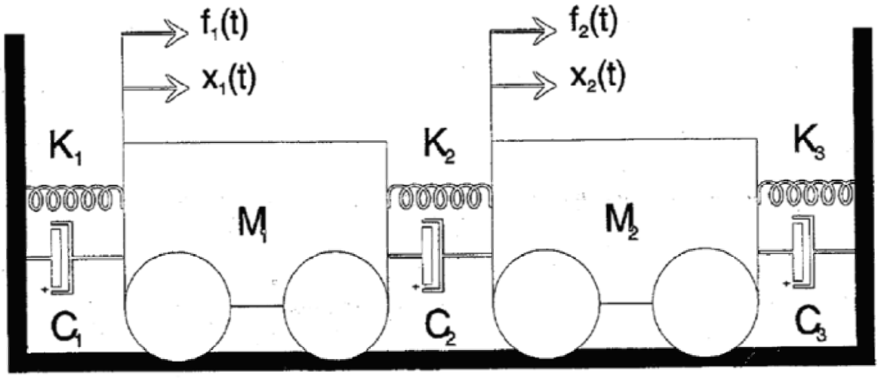
Execution of the above code gives the following results:





Exercise 1.

For the system, as shown in Figure below tune the parameters of the matrices *M, C* and *K* that the first natural frequency is slightly below 5 Hz, and the second above 50 Hz



Exercise 2.

For the chosen parameters in the Exercise 1, perform a synthesis of FRFs with use of the equation from the example and the equation given below:

$$H(s) = \left. \frac{1}{s^2 \cdot M + s \cdot C + K} \right|_{s=j\omega}$$

Compare results.

Exercise 3.

For the system from Exercise 1 perform a modal analysis by replacing the eigenvalue decomposition with the singular value decomposition. Comment on the obtained results.

Exercise 4.

Solve the system of equations,

$$y = A \cdot x$$

Where: A is a matrix of random coefficients with dimensions 1000 x 1000, and vector y is a random vector with dimensions 1000 x 1. Apply the classical method and the LU method. Compare the times of calculation (tic, toc)