Signals & Systems

Laboratory no. 7 – Least Squares Method

Definition

Least squares method (more appropriate, but unused name: a method of minimum sum of squared errors) - the method of statistical estimation and determination of the trend line based on the data set in the form of pairs of numbers.
Least squares method is used for finding polynomial $h_n(x)$, which will be the most accurate approximation of the function *y(x)*, describing the stochastic process so that:

$$\sum_{i=1}^{N}\left(y_i - h_n\left(x_i\right)\right)^2 = \min \tag{1}$$

where: n – polynomial order; N – number of experiments results, and $n<<N$.
To find the minima of this function by changing the unknown polynomial coefficients $a_{i.}$, the function is differentiate successively with respect to $a_{i.}$ and equal to 0. This leads ultimately to the system of equations by which we determine the unknown coefficients of the polynomial $a_{i.}$

$$\begin{aligned} &S_0 a_0 + S_1 a_1 + \cdots + S_n a_n = T_o \\ &S_1 a_0 + S_2 a_1 + \cdots + S_{n+1} a_n = T_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &S_n a_0 + S_{n+1} a_1 + \cdots + S_{2n} a_n = T_n \end{aligned} \tag{2}$$

where:

$$S_i = \sum_{j=0}^{N} x_j^i \quad and \quad T_i = \sum_{j=0}^{N} x_j^i y_j \ ,$$

Least squares method is usually used in linear regression. For linear regression we demand the minimization of function ?^{2}, which measures the deviation of a given functional dependency from the experimental points. In the case of the linear function $f(x) = ax + b$, function ?^{2} takes the following form:

$$c^2(a,b) = \sum_{i=1}^{N} \frac{\left(y_i - ax_i - b\right)^2}{s_i^2} \tag{3}$$

where s $_i$ is a standard deviation (measurement uncertainty) of the given measuring point (in variable *y*).
To find the minima of this function with respect to parameters a and b., the function is differentiate successively with respect to a and b equal to 0.

$$\frac{\partial c^2}{\partial a} = 0 = -2\sum_{i=1}^{N} \frac{x_i\left(y_i - ax_i - b\right)}{s_i^2} \tag{4}$$

$$\frac{\partial c^2}{\partial b} = 0 = -2\sum_{i=1}^{N} \frac{\left(y_i - ax_i - b\right)}{s_i^2} \tag{5}$$

When the standard deviation (measurement uncertainty) for all measuring points is equal, the regression is called of unweighted regression (classical or first kind), then the standard deviation can be excluded and simplifies in the equations on the coefficients a_i.

EXAMPLE 1

With use of the least squares method find a linear equation that best approximates the data below:

x_i	1 2 3 4
y_i	7 6 4 1

We look for the equation of straight line $y = ax + b$, that minimizes the following function:

$$f(a, b) = ((a \cdot 1 + b) - 7)^2 + ((a \cdot 2 + b) - 6)^2 + ((a \cdot 3 + b) - 4)^2 + ((a \cdot 4 + b) - 1)^2$$

In the first step we will determine the derivatives of the minimized function:

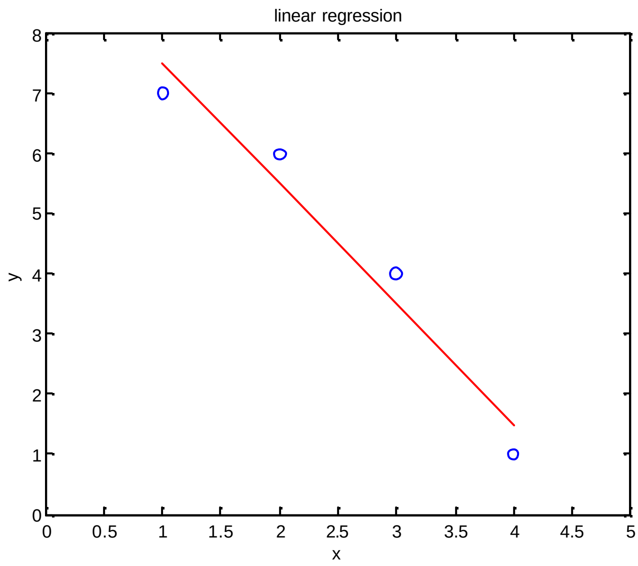
$$\frac{\partial f}{\partial a} = 2 \cdot (a + b - 7) + 2 \cdot (2a + b - 6) \cdot 2 + 2 \cdot (3a + b - 4) \cdot 3 + 2 \cdot (4a + b - 1) \cdot 4 = 2 \cdot (30a + 10b - 35) = 0$$

$$\frac{\partial f}{\partial b} = 2 \cdot (a + b - 7) + 2 \cdot (2a + b - 6) + 2 \cdot (3a + b - 4) + 2 \cdot (4a + b - 1) = 2 \cdot (10a + 4b - 18) = 0$$

From the above system of equations we can calculate the function parameters:

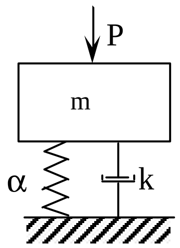
a = - 2,
b = 9,5.

To verify graphically the correctness of the obtained result, the following figure was plotted in Matlab:



EXAMPLE 2

With use of the least squares method find the physical parameters of the system from the figure below:



The equation of motion: $m\ddot{x} + \alpha\dot{x} + kx = P$.
The single sample of the experiment fulfills the equation: $m\ddot{x}_i + \alpha\dot{x}_i + kx_i = P_i$.
The error function e:

$$\sum_{i=1}^{n} \left(m\ddot{x}_i + \alpha\dot{x}_i + kx_i - P_i\right)^2 = e\,.$$

We look for the minimum of the error function with respect to the system parameters m, a, k.

$$\begin{cases} \dfrac{\partial e}{\partial m} = 0 \Rightarrow \displaystyle\sum_{i=1}^{n} 2 \cdot \left(m\ddot{x}_i + \alpha\dot{x}_i + kx_i - P_i\right) \cdot \ddot{x}_i = 0 \\ \dfrac{\partial e}{\partial \alpha} = 0 \Rightarrow \displaystyle\sum_{i=1}^{n} 2 \cdot \left(m\ddot{x}_i + \alpha\dot{x}_i + kx_i - P_i\right) \cdot \dot{x}_i = 0 \\ \dfrac{\partial e}{\partial \alpha} = 0 \Rightarrow \displaystyle\sum_{i=1}^{n} 2 \cdot \left(m\ddot{x}_i + \alpha\dot{x}_i + kx_i - P_i\right) \cdot x_i = 0 \end{cases}$$

We reorganize the equations:

$$\begin{cases} m\sum_{i=1}^{n}\ddot{x}_i^2+\alpha\sum_{i=1}^{n}\ddot{x}_i\dot{x}_i+k\sum_{i=1}^{n}\ddot{x}_i x_i-\sum_{i=1}^{n}P_i\ddot{x}_i=0 \\ m\sum_{i=1}^{n}\ddot{x}_i\dot{x}_i+\alpha\sum_{i=1}^{n}\dot{x}_i^2+k\sum_{i=1}^{n}\dot{x}_i x_i-\sum_{i=1}^{n}P_i\dot{x}_i=0 \\ m\sum_{i=1}^{n}\ddot{x}_i x_i+\alpha\sum_{i=1}^{n}x_i\dot{x}_i+k\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}P_i x_i=0 \end{cases}$$

$$\begin{bmatrix} \sum_{i=1}^{n}\ddot{x}_i^2 & \sum_{i=1}^{n}\ddot{x}_i\dot{x}_i & \sum_{i=1}^{n}\ddot{x}_i x_i \\ \sum_{i=1}^{n}\ddot{x}_i\dot{x}_i & \sum_{i=1}^{n}\dot{x}_i^2 & \sum_{i=1}^{n}\dot{x}_i x_i \\ \sum_{i=1}^{n}\ddot{x}_i x_i & \sum_{i=1}^{n}\dot{x}_i x_i & \sum_{i=1}^{n}x_i^2 \end{bmatrix}\cdot\begin{bmatrix} m \\ \alpha \\ k \end{bmatrix}=\begin{bmatrix} \sum_{i=1}^{n}P_i\ddot{x}_i \\ \sum_{i=1}^{n}P_i\dot{x}_i \\ \sum_{i=1}^{n}P_i x_i \end{bmatrix}$$

After the above steps we obtain a simple matrix equation **X a** = **P**, which can be solved numerically using the Matlab software.

Exercise 1.

Build a model of the mechanical system of Example 2 in Simulink for the following parameters:
m = 2,
c = 0,3,
k = 1000

Exercise 2.

Simulate the system (using the white noise or polyharmonic input signal) and store the time histories of the $\ddot{x}$, $\dot{x}$ and x.

Exercise 3.

Based on the equations derived in Example 2 and the time histories of the $\ddot{x}$, $\dot{x}$ and x, identify the parameters of the system: *m, a, k*

Compare the search and identified parameters of the system. (If the parameters are the same - the calculations are correct, if different, find a bug in the program).

Exercise 4.

Add a noise to the signals $\ddot{x}$, $\dot{x}$ and x. The amplitude of the noise should amount the 0.1 of the noised signals amplitude. Again identify and compare the parameters of the system.